

‘Cusp’ solutions in Gauss–Bonnet gravity

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Einstein-dilaton-Gauss–Bonnet gravity is investigated on existence of solutions with mild singularities, not shielded by the event horizons. These still may have sense since presumably such singularities will be smoothed by corrections to Einstein theory from quantum gravity/string theory. We show that gravity with the first-order correction, the Gauss–Bonnet term, gives rise to special types of singularities, which we call ‘cusps’, with $1/2$ -th and $1/3$ -th powers in series expansion. The full space then can be split onto several cusps with classically impenetrable borders, and/or flat asymptotic.

I. GAUSS–BONNET GRAVITY

Gauss–Bonnet term provides a non-trivial contribution to the Einstein equations only in dimensions $D > 4$, while in $D = 4$ gravity one should include interaction with the dilaton field. The solution without the GB correction is a standard Schwarzschild metrics with constant dilaton (zero dilaton charge is ensured by the ‘no-hair’ theorem). Adding the GB term gives rise to a non-trivial dilaton configuration with two different types of asymptotically flat solutions: the first is the black hole, and the second contains a naked singularity. Actually the BH solution also contains singularity, but this is shielded by the event horizon.

Solutions with naked singularities are usually not in a favor, therefore subsequent investigations were produced only for the black hole branch. But now let’s turn back to our motivation of adding a higher order curvature corrections to the action. It is because of our belief in some fundamental quantum theory we add the GB term. Therefore when we get a ‘singularity’ this just means that we moved outside the semi-classical limit with first-order corrections, but the ‘real’ quantum theory should deal with the growing curvature and energy density in some way. This is why the domain of initial conditions with high energy density should not be forgotten in favor of the domain of initial conditions with the event horizon. So in our work we will explore the family of solutions with GB corrections in more details.

Again, one should mention that since the GB term is motivated by string theory, the corresponding calculations should be produced rather in the string frame than in the Einstein frame, and the transformation of the Gauss–Bonnet term between these frames generates a very complicated additional term. It was shown that not all solutions known in the Einstein frame are reproduced in the string frame [1] in the context of the dilaton black holes with Gauss–Bonnet corrections. For example, the family of singular solutions, obtained by [2] as far as we know was not found in the string frame.

Therefore in our investigation we will provide two parallel calculations: for the usual Einstein-dilaton-Gauss–Bonnet action (to extend known results)

$$\mathcal{S}^{(E)} = \frac{1}{16\pi} \int (R - (\partial_\mu \ln S)^2 / (2a^2) + \alpha S \mathcal{R}_{GB}^2) \sqrt{-g} d^4x, \quad (1)$$

and for the stringy version of the action

$$\mathcal{S}^{(str)} = \frac{1}{16\pi} \int (R + (\partial_\mu \ln S)^2 / a^2 + \alpha \mathcal{R}_{GB}^2) S \sqrt{-g} d^4x. \quad (2)$$

Note that in [1] the first version was called a ‘truncated’ one. We shall use another notation: the first version will be marked as “(E)” (EGBD), and the second version as “(str)” (SEGBD). Along with the Ricci scalar curvature R , it contains the dilaton field $S = e^{2a\phi}$ and the Gauss–Bonnet term \mathcal{R}_{GB}^2 . The Euler density of the GB term has the following form:

$$\mathcal{R}_{GB}^2 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}. \quad (3)$$

The action is written in Plank units with $\hbar = c = G = 1$, and it contains two parameters: a which is a dilaton coupling constant and α which is a ‘correction’ parameter in sense that GB term is treated as a correction to the Einstein’s action. Both parameters are supposed to be positive.

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The static spherically symmetric spacetime can be described by the following metrics:

$$ds^2 = -w(r)\sigma(r)^2 dt^2 + \frac{dr^2}{w(r)} + \rho(r)^2 d\Omega_2^2, \quad (4)$$

which contains three functions depending only on the radial coordinate, but one of them can be gauged away by the rescaling of r .

One can transform the action in the string frame (2) to the Einstein's frame by the conformal transformation $g_{\mu\nu}^{(str)} = S^{-1} g_{\mu\nu}^{(E)}$:

$$\begin{aligned} \mathcal{S}_g^{(str)} &= \int R[g_{\mu\nu}^{(E)} S^{-1}] S \sqrt{-g^{(E)} S^{-4}} d^4x = \\ &= \mathcal{S}_g^{(E)} - 3/2 \int (\partial_\mu \ln S)^2 \sqrt{-g^{(E)}} d^4x. \end{aligned} \quad (5)$$

In string theory one has $a = 1$, therefore $\mathcal{S}_g^{(str)} + \mathcal{S}_d^{(str)} = \mathcal{S}_g^{(E)} + \mathcal{S}_d^{(E)}$, where \mathcal{S}_d is a dilaton kinetic term. So in the case $a = 1$ the standard Einstein-dilaton action without GB corrections is identical in both frames up to the conformal transformation. For arbitrary a the transformation from the string frame to the Einstein frame produces the action with additional kinetic term.

The transformation of the GB term under the conformal transformations is rather complicated. But we can just write the GB-term in the non-transformed frame and substitute the metric functions as $w^{(str)} = S w^{(E)}$, $\sigma^{(str)} = S^{-1} \sigma^{(E)}$, $\rho^{(str)} = S^{-1/2} \rho^{(E)}$. This will allow us to express both versions of the action in one frame, where the high-order curvature correction terms will be different.

Since the GB term is a total derivative:

$$\sqrt{-g} \mathcal{R}_{GB}^2 = \Lambda'_{GB}, \quad \text{where} \quad \Lambda_{GB} = \frac{4}{\sigma} (w\sigma^2)' (w\rho'^2 - 1), \quad (6)$$

it enters the lagrangian like $-\alpha S' \Lambda_{GB}$. The expression above is correct for the $g_{\mu\nu}^{(str)}$ metric functions in the string frame and for the $g_{\mu\nu}^{(E)}$ metric functions in the Einstein frame. Therefore the GB part of the action $\mathcal{S}^{(str)}$ in the Einstein frame will enter also like $-\alpha S' \tilde{\Lambda}_{GB}$, but with the different expression for the Λ_{GB} -term which now reads as

$$\tilde{\Lambda}_{GB} = \frac{4S}{\sigma} \left(\frac{w\sigma^2}{S} \right)' \left[Sw \left(\frac{\rho}{\sqrt{S}} \right)'^2 - 1 \right], \quad (7)$$

where the metric functions were substituted by the transformed ones. This expression can be split onto the same Λ_{GB} -term when the derivatives do not act on the dilatonic function and the terms which are proportional to its derivatives:

$$\tilde{\Lambda}_{GB} = \Lambda_{GB} + \Delta \Lambda_{GB}, \quad \text{where} \quad \Delta \Lambda_{GB} = \sum_{n=2}^4 \Lambda_n (\ln S)^n. \quad (8)$$

One can see that the difference between two forms of the GB term is significant only when dilaton demonstrates the exponential growth, and it vanishes if asymptotically $S \rightarrow \text{const}$.

Now we can obtain the one-dimensional effective EDGB lagrangian:

$$\mathcal{L}^{(E)} = \frac{\rho'(\sigma^2 w \rho)'}{2\sigma} - \frac{\sigma w \rho^2 S'^2}{8a^2 S^2} - \frac{\alpha S'}{\sigma} (w\sigma^2)' (w\rho'^2 - 1). \quad (9)$$

And the lagrangian of the stringy model reads as

$$\mathcal{L}^{(str)} = S \left(\frac{\rho'(\sigma^2 w \rho)'}{2\sigma} + \frac{\sigma}{2} + \frac{\sigma w \rho^2 S'^2}{4a^2 S^2} \right) + \frac{S'(w\sigma^2 \rho^4)'}{4\sigma \rho^2} - \frac{\alpha S'}{\sigma} (w\sigma^2)' (w\rho'^2 - 1). \quad (10)$$

The first solution for the Einstein DGB system was obtained by Kanti et. al. [2] where they obtained the BH-branch of solutions with horizon r_h and another branch with an unshielded singularity r_x where square of the Riemann tensor diverges like $(r - r_x)^{-1}$. Then, Alexeyev et. al. [3] explored the behavior of the solutions under horizon r_h and revealed that then r_s point becomes a “turning point” singularity from which two branches of solutions arise. The first one goes outward from r_s to r_h and the second one moves further into the black hole from r_s to the singularity r_x . In [4] the dynamical analysis was produced and it was shown that the only regular solutions with flat asymptotic are the black holes. In paper [1] the authors reproduced the BH solutions in the string frame and revealed the difference between EDGB and SEDGB models, but they did not investigate non-BH solutions in details, just mention that sometimes there is not a BH, but a singularity.

II. CUSP SINGULARITIES

We treat cusp as a mild singularity of the metrics, where the dilaton function and metric components are regular and non-vanishing, but their second derivatives are singular.

In what follows we will use the shifted radial variable $x = r - r_s$ for convenience. In the vicinity of the singularity the metric and dilaton functions can be written as expansions by powers of x :

$$w = \sum_{n=0}^{\infty} w_{n/z} x^{n/z}, \quad \rho = \sum_{n=0}^{\infty} \rho_{n/z} x^{n/z}, \quad S = \sum_{n=0}^{\infty} p_{n/z} x^{n/z}, \quad (11)$$

with $z = 2, 3, \dots$ parameterizing the order of singularity. First, we will investigate the case $z = 2$, which corresponds to the turning point of the metrics. To provide the regularity of the first derivatives we impose the condition $w_{1/2} = \rho_{1/2} = S_{1/2} = 0$. In this case the curvature has $1/\sqrt{x}$ singularity at the cusp.

Substituting the expansions in to the equations of motion we find that starting from the second order (i.e. $n/2 \geq 2$) the equations are linear by the $n/2$ -th coefficients, expressing them through the previous coefficients. Unfortunately, in the first order (to be precious $3/2$ order) we obtained the system for the $p_{3/2}$ and $q_{3/2}$ which contains polynomials of orders higher than five. Therefore we are unable to provide explicit expansions. Nevertheless, this system can be solved numerically with any required precision when substituting quantities instead of free parameters of the expansions.

The free parameters of the expansions are the parameters of the configuration (a, α) , the dilaton value on the cusp surface, p_0 , the radius of the cusp, ρ_0 , the angular deficit of the cusp which can be expressed by ρ_1 and the time component of the metrics, w_0 . But the last two parameters can be combined into one $u = w_0 \rho_1^2$, while the remaining degree of freedom will be responsible only for the scale transformations. In the case of flat asymptotical metrics the GB term vanishes faster then the curvature term, for which it is well known that the asymptotic should satisfy the relation $w_{\infty} \rho_{\infty}^2 = 1$, while separately w_{∞} and ρ'_{∞} can be not equal to unity. Therefore the parameter u can be treated as a deviance of the cusp metrics from the flat asymptotic. Also the dilaton parameter p_0 can be excluded by the appropriate rescaling of the dilaton function and the GB parameter $\alpha \rightarrow \alpha p_0$. In what follow we will set $p_0 = 1$, so that the effective GB parameter αp_0 will coincide with the initial parameter α in the lagrangian.

A. $z=2$ cusp

The most interesting solutions are those which interpolate between the cusp and the flat asymptotic (Fig. 1), and numerical computations confirmed their existence. Other solutions interpolate between two cusps of different radii (Fig. 2). The actual parameter space is four-dimensional (a, α, ρ_0, u) but it can be schematically described in the following way. The key role in producing the asymptotically flat solutions belongs to the parameter u . It appeared that the deviance from the asymptotic should be small enough, but non-vanishing. There is a forbidden gap around $u = 1$, and two permitted bands on each side of it. The decrease of a, α and the increase of ρ_0 causes the stretching of the allowed bands, while the forbidden gap also enlarges above $u = 1$ and slightly modifies below $u = 1$. Conversely, the increase of a, α and the decrease of ρ_0 causes the contraction of the both permitted bands and the forbidden gap above $u = 1$. To be precise, these bands are not solid, they possess stripy structure with narrow forbidden gaps. But the detailing seems to be abundant for practical purposes.

One can conclude that all conditions which stimulate existence of asymptotically flat solutions can be understood as relative decrease of the contribution of Gauss–Bonnet corrections in comparison with the classical Einstein-dilaton gravity terms. This can be seen from the table which follow. Data in each table starts from the point $a = \alpha = \rho_0 = 1$ and then one of the parameters varies in the direction which enlarges the allowed bands for u .

(a, α, ρ_0)	allowed band		allowed band	
(1, 1, 1)	0.93	0.99	1.02	1.11
(0.5, 1, 1)	0.77	0.99	1.09	1.22
(0.33, 1, 1)	0.65	0.98	1.1	1.34
(1, 0.5, 1)	0.89	0.98	1.06	1.23
(1, 0.33, 1)	0.75	0.94	1.11	1.34
(1, 1, 2)	0.46	0.92	1.16	1.55
(1, 1, 3)	0.6	0.86	1.54	2.5

One can compute the ADM mass and the dilaton charge of the configuration using the asymptotic obtained numerically. It appears that the dilaton charge decreases and the ADM mass increases when the parameter u deviates

from unity. The dilaton charge is negative, while the mass is positive on the lower band and negative for on upper band.

There is another interesting branch of solutions for the considerable deviance from the asymptotic with $u \sim 0.6$. With growing x one obtains solutions with the metrics reaching the flat asymptotic, while the dilaton function grows exponentially. Still these solutions are singular, but the point of singularity moves to infinity with the decrease of α . Actually the plot of the dilaton function looks like that of the scale factor in the inflationary scenario with fast acceleration period turning then into slow deceleration. With the significant decrease of α this branch disappears since for sufficiently small α the lower allowed band of regular solutions reaches the point $u \sim 0.6$.

B. $z=3$ cusp

One can choose the value $z = 3$, i.e. the expansions near cusp will contain powers of $n/3$. In this case the condition of regularity of the first derivatives does not hold, but there is a $w_{2/3}x^{1/3}$ term. The scalar curvature possess stronger singularity $x^{-4/3}$ instead of the square root singularity as it was for $z = 2$.

Unlike the previous case, the coefficients of the expansions can be found explicitly, but they are rather complicated. It turned out that there are two branches of the expansions depending on whether $\rho_{4/3}$ vanishes or not. Moreover, each branch splits into two other branches which are identical up to the transformation $x^{n/z} \rightarrow (-1)^n x^{n/z}$, which defines the ‘direction’ where the solution spreads. The parameter space contains five parameters, but again p_0 and w_0 can be scaled away. So the effective parameter space (a, α, ρ_0) is three-dimensional, and there is no independent parameter like u in previous case.

Surprisingly, we did not found any regular solution with flat asymptotic numerically. Each solution ends by the square-root cusp considered in previous section as shown on (Fig. 3). Nevertheless, there exist the same branch of solutions with the exponentially growing dilaton function when α decreases. But now the absence of regular solutions does not bound this branch from below. So with $\alpha \rightarrow p_0$ one has $S \rightarrow \infty$ and $w, \rho' \rightarrow \text{const}$ what gives us the flat metrics. The point of singularity moves to infinity. Mention that there is no cusp for $\alpha = 0$, since the terms with non-integer powers of x vanish in the expansions both for $z = 2$ and $z = 3$.

III. CONCLUSION

We investigated the class of singular solutions to Gauss–Bonnet dilaton gravity system. First, we mentioned already known singular solution with $1/2$ -th powers expansion near singularity. Such solutions can provide flat asymptotic, but also they can demonstrate from-cusp-to-cusp behavior, where the entire space can be split onto several regions by cusps which are classically impenetrable. Then we showed the existence of a new singular solution to Gauss–Bonnet dilatonic gravity, containing $1/3$ -th powers near the singular point. It can not produce flat asymptotic and always spreads from one singularity to another. But it improves the inner region in the sense that one has flat space between cusps.

Still there is an open question if there are other cusps, probably even with not a power-series expansions. Such nonlinear differential equations can provide a very diverse picture. We investigated $1/4$ - and $1/5$ -th powers and found no cusp solutions. Another question which we explored was the difference between classical and string versions of dilatonic gravity with curvature corrections. It appeared that qualitatively the picture is almost the same.

One can ask about applications of cusp solutions. Some of them can be interesting in view of the brane-world scenarios, where the singularity is moved to the bulk [5–7]. Finally, if the black holes were treated as some unnatural solutions for many years, and now we have several dozens of candidates to them, probably another even more strange configurations will arise in future astrophysics.

This work was supported by Dynasty foundation and the RFBR grant 08-02-01398-a.

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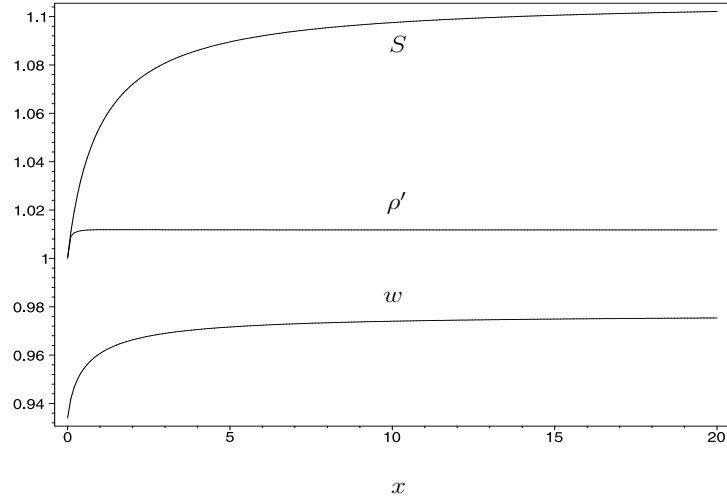


FIG. 1: $z = 2$: from cusp to Minkowski asymptotic.

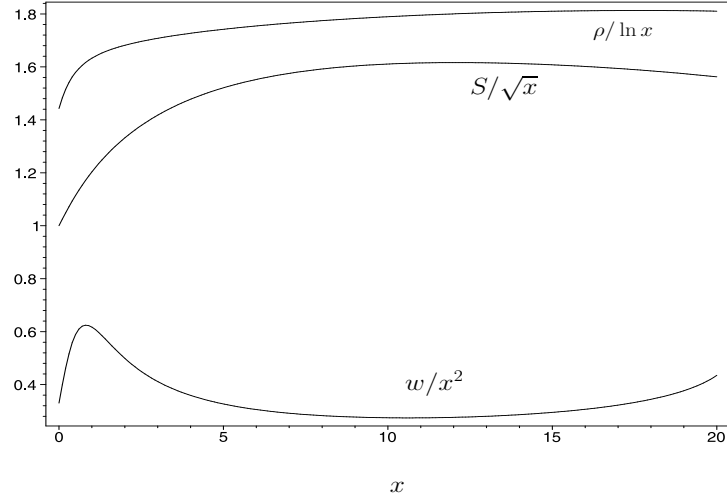


FIG. 2: $z = 2$: from cusp to cusp.

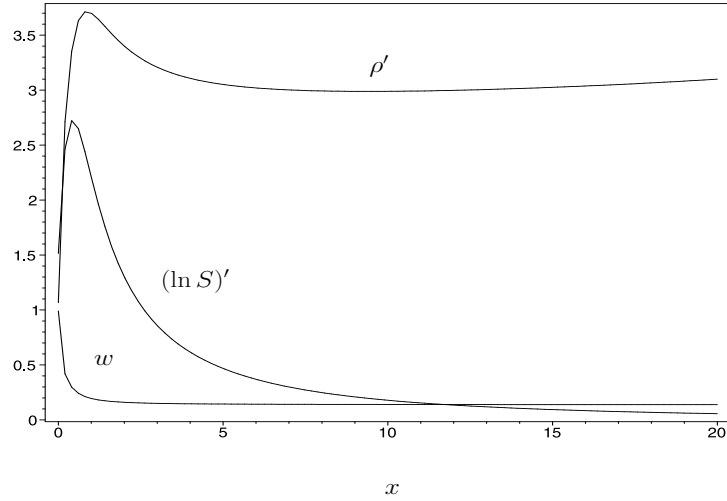


FIG. 3: $z = 3$: from cusp to cusp $z = 2$ through the flat middle area.